

AD-A141 519

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND RELATED
PROBLEMS OF PADE APP. (U) COLUMBIA UNIV NEW YORK DEPT
OF MATHEMATICS D V CHUDNOVSKY ET AL. DEC 83

1/1

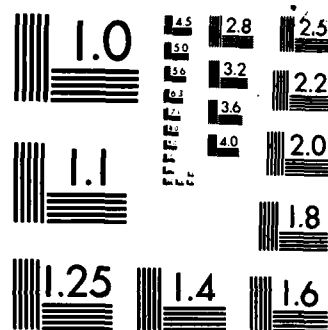
UNCLASSIFIED

AFOSR-TR-84-0291 AFOSR-81-0190

F/G 12/1

NL





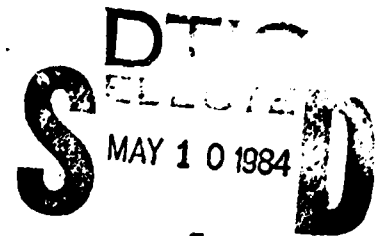
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AFOSR-TR. 84-0291

Progress Report

Nonlinear Partial Differential Equations and
Related Problems of Padé Approximations

by D.V. Chudnovsky and G.V. Chudnovsky



During the period of the grant we continued our studies of one-four dimensional classical and quantum lattice and continuous completely integrable systems. Our technique is based on the method of Bäcklund transformations and their algebraic, geometric and arithmetic properties. Methods of Bäcklund transformations were successfully used in the study of Padé approximations. One of our significant achievements during the last year was the solution of the key problem on the almost perfectness of Padé approximations to solutions of linear differential equations.

I. Nonlinear differential equations in dimensions one, two and three with the complete integrability property.

The investigators studied three dimensional lattice and continuous models with the complete integrability property using Bäcklund transformations (BTs) algebra [1], [2], [3], [4]. Our approach, outlined in [1], is based on topological properties of symplectic structures and analytic deformations

Approved for public release;
distribution unlimited.

84 04 24 001

AD-A141 519

DTIC FILE COPY

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE					
4. PERFORMING ORGANIZATION REPORT NUMBER(S)			5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 84-0291		
6a. NAME OF PERFORMING ORGANIZATION Columbia University		6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research		
6c. ADDRESS (City, State and ZIP Code) Department of Mathematics New York NY 10027			7b. ADDRESS (City, State and ZIP Code) Directorate of Mathematical & Information Sciences, Bolling AFB DC 20332		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-81-0190		
8c. ADDRESS (City, State and ZIP Code) Bolling AFB DC 20332			10. SOURCE OF FUNDING NOS.		
			PROGRAM ELEMENT NO. 61102F	PROJECT NO. 2304	TASK NO. A4
11. TITLE (Include Security Classification) NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS AND RELATED PROBLEMS OF PADE APPROXIMATIONS					
12. PERSONAL AUTHOR(S) David V. Chudnovsky and Gregory V. Chudnovsky					
13a. TYPE OF REPORT Interim		13b. TIME COVERED FROM 30/6/82 TO 29/6/83		14. DATE OF REPORT (Yr., Mo., Day) DEC 83	
				15. PAGE COUNT 14	
16. SUPPLEMENTARY NOTATION					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB. GR.			
19. ABSTRACT (Continue on reverse if necessary and identify by block number) During the period of the grant the investigators continued their studies of one-four dimensional classical and quantum lattice and continuous completely integrable systems. Their technique is based on the method of Backlund transformations and their algebraic, geometric and arithmetic properties. Methods of Backlund transformations were successfully used in the study of Pade approximations. One of the significant achievements during the last year was the solution of the key problem on the almost perfectness of Pade approximations to solutions of linear differential equations.					
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS <input type="checkbox"/>			21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL Dr. Robert N. Buchal			22b. TELEPHONE NUMBER (Include Area Code) (202) 767- 4939		22c. OFFICE SYMBOL NM

DD FORM 1473, 83 APR

EDITION OF 1 JAN 73 IS OBSOLETE.

84 04 24 001

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE

of them) expressed in terms of commutativity and associativity axioms for S-matrices associated with physical systems. This topological approach is used for classical and quantum(operator valued) systems. Commutativity and associativity axioms take the form of factorization equations on elements of S-matrices, and are equivalent to algebraic identities satisfied by BTs of solutions of corresponding nonlinear differential equations. These algebraic identities satisfied by BTs take the form of universal discrete completely integrable equations [1], [3]. We present two of these universal equations in dimensions three and four respectively:

$$\begin{aligned}
 & (\lambda_2 - \lambda_3) \sigma_{\overline{B}_1} \cdot \sigma_{\overline{B}_2} \cdot \overline{B}_3 + (\lambda_3 - \lambda_1) \sigma_{\overline{B}_2} \cdot \sigma_{\overline{B}_3} \cdot \overline{B}_1 + \\
 (1.1) \quad & + (\lambda_1 - \lambda_2) \sigma_{\overline{B}_3} \cdot \sigma_{\overline{B}_1} \cdot \overline{B}_2 = 0; \\
 & (\lambda_1 - \lambda_4) (\lambda_2 - \lambda_3) \sigma_{\overline{B}_1} \cdot \overline{B}_4 \cdot \sigma_{\overline{B}_2} \cdot \overline{B}_3 + (\lambda_3 - \lambda_1) (\lambda_2 - \lambda_4) \sigma_{\overline{B}_1} \cdot \overline{B}_3 \cdot \sigma_{\overline{B}_2} \cdot \overline{B}_4 \\
 (1.2) \quad & + (\lambda_2 - \lambda_1) (\lambda_4 - \lambda_3) \sigma_{\overline{B}_1} \cdot \overline{B}_2 \cdot \sigma_{\overline{B}_4} \cdot \overline{B}_3 = 0,
 \end{aligned}$$

where $\sigma_{\overline{B}_i}$ is an elementary BT corresponding to the addition

of an apparent singularity at $\lambda = \lambda_i$ to σ , see [1]. In a

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)
NOTICE OF TECHNICAL INFORMATION

This technical information has been reviewed and approved for release on 11 APR 1980-12.

Distribution is unlimited.

MATTHEW J. KERPER

Chief, Technical Information Division

particular case, when completely integrable systems are associated with second order linear differential (or difference) spectral problems, $\sigma_{\bar{B}}$ has the form $\sigma_{\bar{B}} = W(\psi_1, \dots, \psi_k) / \det(\lambda_i^{j-1})_{i,j=1,\dots,k}$ for k eigenfunctions ψ_i corresponding to eigenvalues λ_i ; with BT data $\bar{B} = (\lambda_1, \dots, \lambda_k)$;

and a Wronskian (continuous or discrete) of functions ψ_1, \dots, ψ_k . Discrete equations (1.1), (1.2) take the form of the law of addition, when BTs are represented as translation operators in the auxiliary space of infinitely many variables x_n : $n = 1, 2, \dots$ (being Newton's symmetric functions), with BT action $\sigma_{B_i} \stackrel{\text{def}}{=} \exp \left\{ \sum_{n=1}^{\infty} \lambda_i^{-n} \frac{\partial}{\partial x_n} \right\} \circ \sigma$ or

$\sigma_{B_i}(\vec{X}) = \sigma(\vec{X} + \vec{\lambda}_i)$. The equations (1.1), (1.2) in this

form are equivalent to the laws of addition on curves (of genus $g \leq \infty$), and are crucial in the complete solution of the Schottky problem of the determination of Jacobian varieties among all Abelian varieties.

Discrete universal equations (1.1) generate in various limits and reductions the Kadomtzev-Petviashvili' (KP) equation and various other well known two dimensional completely integrable systems, see [5]. We have proved in [1], [4] that all multicomponent (operator) two dimensional completely integrable systems of isospectral deformation nature are algebraic reductions of the universal equation (1.1). Moreover, particular algebraic reductions of (1.1) that determined two-dimensional multicomponent completely integrable systems were completely described in [4] using infinite dimensional algebras arising from particular algebras of pseudodifferential operators. Recently we were able to prove that any multicomponent (matrix, operator) three dimensional systems with the complete integrability property also arise from systems (1.1) with

a particular representation of the algebra of BTs. These three dimensional matrix systems include any system of nonlinear p.d.e. that arises as a commutativity condition $[L_1, L_2] = 0$, where L_1, L_2 are linear differential operators in $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ with matrix (operator) coefficients (and are of the first order in $\frac{\partial}{\partial y}, \frac{\partial}{\partial z}$). These three-dimensional matrix systems include the general monopole equations for $SU(n)$ or $SO(n)$ cases and, say, the matrix generalizations of the KP equation derived by the authors in [6].

The investigators, together with M. Tabor, succeeded in proving the "Painlevé property" for two dimensional multicomponent isospectral deformation equations including various versions of the matrix nonlinear Schrödinger, mdV and KdV equations [7]. This led to the establishment of new completely integrable many particle systems arising in the pole expansion of meromorphic solutions (as a particular case of Painlevé's expansion). These results were based on the above mentioned reductions to the universal equation [4], [7]. The investigation of the Painlevé property is extended to matrix three dimensional systems. Simultaneously, we arrived at a solution of a new algebraic problem of the determination of all exponents of various branches of the Painlevé expansions of solutions of various flows committing with a given completely integrable one (already a nontrivial problem for the n -th KdV flow).

Painlevé equations without movable critical points were found by the investigators to be of utmost importance

in the solution of the functional Mordell problem [8].
(Here the Painlevé VI and the so-called Painlevé equation of
type VIII are crucial).



A1

II. Padé type approximation methods and their applications in mathematics and mathematical physics.

We are continuing to be active in theoretical and computer studies of Padé approximations to solutions of differential equations. Here, under the generic name of Padé-type approximations, we understand the approximation of functions having convergent or asymptotic power series expansions, locally by rational functions. If we consider approximations in only nonarchimedian metrics ($|t|_{x_0} = \rho^{-m}$ iff $f(x) = \sum a_n(x - x_0)^n$ and $m = \min \{n : a_n \neq 0\}$). Then we obtain classical (multipoint) Padé approximations. Our most spectacular results during the last year concern Padé-type approximations to solutions of linear differential equations. Padé-type approximations in this case are directly connected with the algebraic formalism of the inverse scattering method and Bäcklund transformations [2], [3], [9], [10]. Padé approximations to solutions of linear differential equations are very attractive in view of the possibility of establishing rigorously the convergence of this method for multivalued solutions of differential equations following the monodromy properties. The efficiency of Padé and Padé-type approximations is also demonstrated by the fact that these methods furnish exceptionally good rational approximations to numbers that are values of solutions of differential equations. Sometimes we obtain explicit expressions of the continued fraction expansions of these numbers. For example, we studied new, Padé-type approximations to such functions as $\ln(1 - 1/z)$ at

$z \sim \infty$ and $\frac{1}{\sqrt{2}} \operatorname{arctg} \frac{1}{\sqrt{2}}$. In this way we obtain explicit expressions for the best rational approximations to transcendental numbers such as $\ln 2$ and $\frac{\pi}{\sqrt{3}}$. See the Appendix for explicit three-term recurrences relating rational approximations to $\frac{\pi}{\sqrt{3}}$. These rational approximations give outstanding bounds for measures of diophantine approximations to classical transcendental numbers [8], [11], [12].

Our main result [14], [15] on the theory of Padé approximations itself is the proof of the "almost almost perfectness" of Padé approximations to solutions of arbitrary (algebraic) differential equations. Our result [14], [15] gives a complete solution to Kolchin's problem [13] that for any solution $f(x)$ of an arbitrary (algebraic) differential equation over $\mathbb{C}(x)$ regular at $x = \infty$, any $\epsilon > 0$ and any arbitrary rational function $P(x)/Q(x)$, we have $\operatorname{ord}_x = \infty \left(f(x) - \frac{P(x)}{Q(x)} \right) \leq (2 + \epsilon) \max \{ \deg(P), \deg(Q) \} + C_0(\epsilon, f)$. We want to note that C.F. Osgood recently announced that he too has an effective solution to Kolchin's problems in the case of solutions of linear differential equations.

The general theorem [14], [15] of the investigators shows that Padé approximations are "almost almost perfect." To invoke the notion of perfectness we remind of the structure of the Padé table for $f(x)$. On the (n, m) th place of the Padé table for $f(x)$ at $x = x_0$, we put a rational function $P_n(x)/Q_m(x)$, such that $\operatorname{ord}_{x=x_0} (f(x) - P_n(x)/Q_m(x)) > n+m$. Then the Padé table is divided into squares consisting of identical rational functions. The standard

definition of perfectness (normality) means that each square is of size one. Unfortunately, this natural definition is not universally valid (e.g. it is enough to multiply any $f(x)$ by an appropriate rational function to create squares of an arbitrarily large size). A more reasonable condition is almost perfectness (Mahler, 1935 [16], [8]) meaning that squares in the Padé table have uniformly bounded sizes. This is equivalent to $\epsilon = 0$ in the solution of the Kolchin problem above, while the investigators' solution [14], [15] of the Kolchin problem is only an "almost almost perfectness" statement. The investigators now have the proof of almost perfectness for a large class of linear differential equations, including differential equations of hypergeometric type, and for arbitrary algebraic functions. The only analytic results that existed before were those of Arms and Edrei, see [17], for particular trigonometric functions such as $\cos x$, $\frac{\sin x}{x}$ using the positivity argument.

Our results are generalized to Hermite-Padé approximations to sequences of functions [14], [15]. These results and the methods of algebraic geometry and differential algebra that we use, were already applied by the investigators [15], [18] to study the diophantine approximations of values of solutions of differential equations. In particular, as a solution of Lang's problem we proved the best possible " $2 + \epsilon$ " bound in the measure of irrationality of any value θ of E-function $f(z)$ at rational $z = r$. Here the " $2 + \epsilon$ " bound means that for any $\epsilon > 0$, $|\theta - \frac{p}{q}| > |q|^{-2-\epsilon}$ for rational integers p, q with $|q| \geq q_0(\theta, \epsilon)$. An E-function $f(z)$ is a function $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ with $a_n \in \mathbb{Q}$,

satisfying a linear differential equation with rational function coefficients, and $\theta = f(r)$, $r \neq 0$.

From the point of view of rational approximations, our results for $\epsilon = 0$ imply the boundedness of (degrees of) elements in the continued fraction expansion of solutions of linear differential equations. These results and the solution of the Kolchin problem are the first step in the explicit determination of rates and domains of convergence of rational approximations to solutions of arbitrary differential equations.

Our work was facilitated by use of symbolic algebra manipulation systems SCRATHPAD at IBM and one of the versions SMP(Caltech).

References

- [1] D.V. Chudnovsky, in Group Theoretical Methods in Physics, Lecture Notes Physics, v 180 (1933), 66-89.
- [2] D.V. Chudnovsky, in Problems in Mathematics and Physics, Festschrift for F. Gürsey, Gordon and Breach, 1983(in press).
- [3] D.V. Chudnovsky, G.V. Chudnovsky, Proc. Natl. Aca Sci., USA, 80(1983), pp. 1774-1777.
- [4] D.V. Chudnovsky, G.V. Chudnovsky, Phys. Lett. 93A (1933), 371-374.
- [5] R. Hirota, J. Phys. Soc. Japan, 50(1981), 3785.
- [6] D.V. Chudnovsky, Lecture Notes Physics, v. 126, 1980, 352-416.
- [7] D.V. Chudnovsky, G.V. Chudnovsky, Phys. Lett. 97A(1933), 268-274.
- [8] D.V. Chudnovsky, G.V. Chudnovsky, Padé and Rational Approximations of Systems of Functions and their Arithmetic Applications, Proceedings of the New York Number Theory Seminar, Lecture Notes in Math., 1933(to appear).
- [9] D.V. Chudnovsky, G.V. Chudnovsky, J. Math. Pures et Appl. 61(1932), 1-16.
- [10] G.V. Chudnovsky, in Problems in Mathematics and Physics, Festschrift for F. Gürsey, Gordon and Breach, 1933(in press).
- [11] G.V. Chudnovsky, Proc. Natl. Acad. Sci. USA 80(1933), 3138-3141.
- [12] G.V. Chudnovsky, Arithmetic and Geometry, v. I, Progress in Mathematics, v.35, Birkhauser, Boston, 1933, 61-105.

- [13] E.R. Kolchin, Proc. Amer. Math. Soc. 10(1959), 238-244.
- [14] D.V. Chudnovsky, G.V. Chudnovsky, Proc. Natl. Acad. USA, 80(1983), 5158-5162.
- [15] D.V. Chudnovsky, G.V. Chudnovsky, Padé Approximations to Solutions of Linear Differential Equations and Applications to Diophantine Analysis, Proceedings of the New York Number Theory Seminar, Lecture Notes in Math., 1983(to appear).
- [16] K. Mahler, Composito Math. 19(1968), 95-166.
- [17] W.B. Jones, W.T. Thron, Continued Fractions, Encyclopedia of Mathematics and its Applications, v.11, Addison-Wesley, 1980.
- [18] G.V. Chudnovsky (to appear).

Appendix

We present the explicit expressions for the "good" rational approximations to $\frac{4\pi}{3\sqrt{3}}$, and the three-term recurrence that these approximations satisfy. The explicit formulas are specializations of Padé-type approximations to $\frac{8}{\sqrt{z}} \operatorname{arctg} \frac{1}{\sqrt{z}}$ at $z = 3$. In the rational approximations X_n/Y_n to $\frac{4\pi}{3\sqrt{3}}$, both sequences X_n and Y_n are solutions of the following three-term recurrence:

$$(2.1) \quad A_2(n) Z_{n+2} + A_1(n) Z_{n+1} + A_0(n) Z_n = 0$$

for $Z_n = X_n$ or $Z_n = Y_n$: $n = 0, 1, 2, \dots$. The coefficients $A_0(n)$, $A_1(n)$, $A_2(n)$ are polynomials of degree 9 in n :

$$A_2(n) = -8 \cdot (4n + 7) \cdot (4n + 5) \cdot (4n + 3) \cdot (4n + 1) \cdot (2n + 3) \cdot$$

$$\cdot (n + 2) \cdot (27279n^3 + 52164n^2 + 31511n + 6046);$$

$$A_1(n) = 3 \cdot (4n + 3) \cdot (4n + 1) \cdot (15484624281n^7 +$$

$$122518066482n^6 + 401859218160n^5 +$$

$$706125904254n^4 + 715282318379n^3 +$$

$$415975459648n^2 + 128021157420n +$$

$$16022087856);$$

$$A_0(n) = .54 \cdot (6n + 5)(6n + 1)(3n + 2) \cdot (3n + 1) \cdot$$

$$\cdot (2n + 1)(n + 1) \cdot (27279n^3 + 134001n^2 + 21767n + 117000).$$

The initial conditions for the solutions X_n and Y_n of (2.1) are

$$X_0 = 0, X_1 = 3023; Y_0 = 1, Y_1 = 1250.$$

The explicit expressions for the solution Y_n of (2.1) is

$$Y_n = \sum_{i_1 \leq 3n, i_2 \leq 3n, i_1 + i_2 \leq 4n} \binom{3n}{i_1} \cdot \binom{3n}{i_2} \cdot$$

$$\cdot \binom{2(4n - i_1 - i_2)}{4n - i_1 - i_2} \cdot (-1)^{i_1 + i_2} \cdot 4^{i_2} \cdot 3^{i_1},$$

or

$$Y_n = \sum_{i=0}^{3n} \binom{3n}{i} \cdot \binom{-1/2 + 3n}{4n - i} \cdot 4^{4n-i} \cdot 3^i.$$

Using the approximation X_n/Y_n to $\frac{4\pi}{3\sqrt{3}}$ we obtain the following bound for the measure of irrationality of $\pi/\sqrt{3}$: $|\pi/\sqrt{3} - p/q| > |q|^{-4.81\dots}$ for all rational integers p, q ; $|q| \geq q_0$.

REPROD

FILMED

DANIC